AN INTRODUCTION TO HIERARCHIAL VALVERDE CONSTRUCTIONS

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ABSTRACT. In a 2022 talk, Iván Ongay Valverde gave a special construction on topological formulae on the Projective Hierarchy, in which it proceeds primarily via induction. This construction is now known as a *Hierarchical Valverde Construction*. Valverde constructions are used to prove and show particular projective and σ-projective formulae on the Projective Hierarchy.

1. INTRODUCTION

In his original talk, Valverde proved the following:

Theorem 1.1. *(Valverde) Given $\xi < \omega_1$, every separable metrizable K-$\mathbf{\Sigma}^1_\xi$ and K-$\mathbf{\Pi}^1_\xi$ are projectively σ-projective.*

in which "K" in this context is to denote sets (generally analytic) not included in the Real line, and also:

(1) A space is K-analytic (K-$\mathbf{\Sigma}^1_1$) iff it is the continuous image of a Lindelof Cech-complete space, and
(2) A space is K-analytic iff it is the USCCV image of the irrationals (or ω^ω).

The Projective hierarchy formed using K is called the *K-projective hierarchy*.

Definition 1.1. A space is co-K-analytic (K-$\mathbf{\Pi}^1_{\xi+1}$) iff it is homeomorphic to the Stone-Cech remainder of a K-analytic.

We also use the notion of *USCCV functions*[1], a.k.a. *Upper-Semi Continuous Compact-Valued functions*, defined as $F(x)$ is *USCCV* iff $F(x)$ is compact for all $x \in X$ and F is *Upper-Semi Continuous.* They are important in the fact that they "meta"-generalize compactifications, especially the Stone-Cech Compactification.

Theorem 1.2. *If there exists a USCCV multifunction $F : X \to Y$, then there exists a Stone-Cech Compactification for all subsets of X in Y.*[2]

The K-Projective hierarchy is constructed as:

Definition 1.2. A topological space is K-$\mathbf{\Sigma}^1_{\xi+1}$ (respectively, K-$\mathbf{\Pi}^1_{\xi+1}$), $\xi < \omega_1$, if it is the cont. image of a K-$\mathbf{\Pi}^1_{\xi+1}$ USCCV function from K-$\mathbf{\Pi}^1_{\xi+1}$ to K-$\mathbf{\Sigma}^1_{\xi+1}$ (and vice versa for K-$\mathbf{\Pi}^1_{\xi+1}$).

also,

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[1]This is a mild misnomer, as the original notion was of *multifunctions*.

[2]We only consider compact Hausdorff spaces.

Theorem 1.3. *If there exists a USCCV multifunction $F : X \to Y$, then there exists a Stone-Cech remainder for all subsets of X given that X is K-$\mathbf{\Sigma}^1_{\xi+1}$.*

In fact, one can formulate Definition 1.2 in terms of Stone-Cech remainders, in that a topological space is K-$\mathbf{\Pi}^1_\xi$ if it is homeomorphic to the Stone-Cech remainder of a K-$\mathbf{\Sigma}^1_\xi$. Also, in Theorem 1.3, X need not be K-$\mathbf{\Sigma}^1_\xi$, and can be replaced with K-$\mathbf{\Sigma}^1_\xi$ replaced by K-$\mathbf{\Pi}^1_\xi$. But there are caveats; the Polish space Y given the projection formed from $C \subseteq X \times Y$ must be Suslin, because non-Polishness of Topological spaces (which are preserved using Suslin spaces) negates any possibility of Compactness/Stone-Cech-ness of Y or pretty much C in general under cont. mappings.[3]

Valverde "proved" this in a talk:

Theorem 1.4. *Given a K-$\mathbf{\Pi}^1_\xi$ space X, $\beta X/X$ is K-$\mathbf{\Sigma}^1_\xi$., which is represented with Y.*

We know that if X is K-$\mathbf{\Pi}^1_\xi$, then it must be of the form $\beta Y/Y$. But this results leads to circularity within Theorem 1.4. Therefore, Theorem 1.4 is false.

2. Proof of Theorem 1.1 and commentary

Proof. To start with (the base case), every separable metrizable K-$\mathbf{\Sigma}^1_1$ space is $\mathbf{\Sigma}^1_1$ trivially.[4] For K-$\mathbf{\Pi}^1_1$, per Definition 1.2, it is the continuous image of a USCCV function from K-$\mathbf{\Sigma}^1_1$. Then there exists a βX (consequence of Theorem 1.3) for a topological space X which is K-$\mathbf{\Sigma}^1_1$. Given

$$K\text{-}\mathbf{\Pi}^1_1 \to K\text{-}\mathbf{\Sigma}^1_1 \to \beta X,$$

we have that K-$\mathbf{\Pi}^1_1 \to \beta X$ using USCCV or similar compactification or function, which is defined as *Stone-Cech with Upper-Semi Continuity.*

$$\begin{tikzcd}
K\text{-}\mathbf{\Pi}^1_1 \arrow[d, "\text{USCCV}"'] \arrow[dr, "\text{Stone-Cech with Upper-Semi Continuity}"] & \\
K\text{-}\mathbf{\Sigma}^1_1 \arrow[r, "\text{Stone-Cech}"'] & \beta X
\end{tikzcd}$$

To complete the induction, define a USCCV function from K-$\mathbf{\Pi}^1_1$ to K-$\mathbf{\Sigma}^1_2$ in that such $C \subseteq X \times Y$ is a subset of USCCV. More specifically, there exists a point of a subset of a top. space X for all subsets of X, such that $F : X \to Y$ sends it in a non-continual manner to Y, in $Y \cap X \setminus A$. We call this a *push-up USCCV function.* Also, remark that separable metrizability is required, and comes from $graph(F)$.

[3]Comes from the separability of Polish spaces.

[4]We can also prove this with USCCV, Polish, and Stone-Cech Compactifications.

$$
\begin{tikzcd}
K\text{-}\mathbf{\Pi}^1_{\xi(+\eta)}\text{-space} \arrow[r, "\text{push-up USCCV}"] & K\text{-}\mathbf{\Sigma}^1_{\xi+1}\text{-space} \arrow[dl, "\text{from the graph of USCCV's inv. functions, are } \Pi_{\xi+1}."] \\
\text{All graphs of F are separable measurable} \arrow[u, "\text{from graph(F) being } \mathbf{\Pi}^1_{\xi+1}"] &
\end{tikzcd}
$$

□

2.1. **Commentary.** Hierarchical Valverde Constructions are extremely malleable. Our original proof used graphs of the push-up USCCV's inv. functions, but we could have easily used a different embedding or compactification. Indeed, one can apply this to the Borel Hierarchy or even any hierarchy in general which uses the Sigma-Pi notation. Let us generalize the graph in the Proof of Theorem 1.1:

Definition 2.1. (A boldface Hierarchical Valverde Construction.)

$$
\begin{tikzcd}
\mathbf{\Pi}^1_1\text{-space} \arrow[r, "\text{cont or semi-cont.}"] & \mathbf{\Sigma}^1_2\text{-space} \arrow[dl, "\text{property of the cont or semi-cont. being } \mathbf{\Pi}^1_1"] \\
\text{preserved property under cont or semi-cont.} \arrow[u, "\text{preserved property being } \mathbf{\Pi}^1_1"] &
\end{tikzcd}
$$

and this can be extended to lightface spaces, non-Analytic spaces, and even sets in the Levy Hierarchy. Especially for the Levy Hierarchy, elementary embeddings can be used. We use Hierarchical Valverde Cosntructions to show (inductively) that a certain property holds for all ranks in the Projective hierarchy.

Example 1. Take a subset $\mathcal{A}$ of the Real line[5] that is co-analytic ($\mathbf{\Pi}^1_1$), and suppose that there is a homeomorphism $f : \mathcal{A} \to \mathcal{B}$. To construct $\mathcal{B}$, let the Polish space Y be the Real line. Then $\mathcal{B} \subset \mathbb{R}^2 \restriction \mathbf{\Pi}^1_1$ ($\mathbb{R}^2$ resitricted to co-analytic spaces may also be expressed as $\mathbb{R}^2 \restriction \mathbb{R} \setminus \mathcal{A}$). Then $\mathcal{B}$ is $\mathbf{\Sigma}^1_2$. The $\mathbf{\Pi}^1_1$-property which is "contained" in f is homeomorphisms of (generally complements of) subsets of $\mathcal{A} \times \mathcal{B}$.

$$
\begin{tikzcd}
\mathcal{A} \arrow[rr, "f"] & & \mathcal{B} \arrow[dl, "\mathbf{\Pi}^1_1"] \\
& \text{homeomorphisms of } C \arrow[ul, "\mathbf{\Pi}^1_1\text{-preservation}"] &
\end{tikzcd}
$$

[5]A separable, metrizable subspace of the Real Line is technically $\mathbf{\Delta}^1_1$.

and one can of course show, via induction on $\mathbf{\Pi}^1_\xi$, that there exists a homeomorphism

$$f_\xi : \mathbf{\Pi}^1_\xi\text{-real-line-based spaces} \to \mathbf{\Sigma}^1_{\xi+1}\text{-real-line-based spaces}$$

in which a "real-line-based space" is a space constructed recursively using the Projective hierarchy from either $\mathbb{R}$ itself or a subspace of $\mathbb{R}$.

Lemma 2.0.1. *Every separable metrizable subspace of the Real line is projectively σ-projective.*

In fact, one can easily generalize the notion of a "real-line-based space" constructed via recursion on the Projective hierarchy to other (sub-)spaces. In Definition 2.1, the property need not even be $\mathbf{\Pi}$; it is allowed to be $\mathbf{\Sigma}$, although $\mathbf{\Pi}$ allows for a more restrictive function. Take, for example, a certain pathological subset of $\mathcal{A}$ which is non-one-point-compactible, but the homeomorphism (or function in general) is such that $f : X \to X^\infty$, denoting the one-point compactification. f being or having a $\mathbf{\Pi}$-property would "remove" this pathological subset, in that it is not a member of X-complement or similar.

One can also generalize Example 1 to other Topological fields.

Question 1. How much do bases of Topological spaces (when constructed via recursion of the Projective hierarchy) relate to the basis or top. space of the base layer of the Projective hierarchy?

3. Acknowledgements

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